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Finite analogues of non-Euclidean spaces and Ramanujan graphs

Eiichi Bannai, Osamu Shimabukuro, Hajime Tanaka

Graduate School of Mathematics, Kyushu University, Hakozaki 6-10-1, 812-8581 Fukuoka, Japan

Dedicated to the memory of J.J. Seidel

Abstract

This is a companion paper of "Finite Euclidean graphs and Ramanujan graphs" by the same authors. Finite analogues of the Poincaré upper half plane, i.e., finite upper half plane graphs, were studied by many researchers, including Terras, Evans etc. Finally, it was proved by combining works of A. Weil, P. Deligne, R. Evans, H. Stark, N. Katz, W. Li and many others, that the finite upper half plane graphs of valency $q+1$ over the finite field $\mathbb{F}_q$ are all Ramanujan graphs. In this paper, we obtain further examples of families of Ramanujan graphs, by using previous works on association schemes and the calculations of their character tables, which are in some sense analogues of the finite upper half planes over finite fields, i.e., finite versions of non-Euclidean spaces. A key observation is that in many (but not all) cases, we can obtain a sharper esitimate $|\theta| \leq 2\sqrt{q-2}$ on eigenvalues, instead of the original $|\theta| \leq 2\sqrt{q}$, which was proved by Katz. We combine this observation with the ideas of controlling association schemes and the Ennola type dualities, in our previous papers such as Bannai–Hao–Song (J. Combin. Theory Ser. A 54 (1990) 164), Bannai–Hao–Song–Wei (J. Algebra 144 (1991) 189), Bannai–Kwok–Song (Mem. Fac. Sci. Kyushu Univ. Ser. A 44 (1990) 129), Kwok (Graphs Combin. 7 (1991) 39), Tanaka (Master Thesis (2001)), Tanaka (European J. Combin. 23 (2002) 121) and many others. At the end, we remark that for each fixed valency $k \geq 3$ there are only finitely many distance-regular Ramanujan graphs of valency k.

Introduction

The finite upper half planes over finite fields $\mathbb{F}_q$ have been studied by many authors. When q is odd, they are defined as follows (see Terras [19, Chapter 19] for details). Namely, let δ be a non-square element of $\mathbb{F}_q^{\times}$, and we define the *finite upper half plane* H_q by

$$H_q = \{z = x + y\sqrt{\delta} \mid x, y \in \mathbb{F}_q, y \neq 0\} \subset \mathbb{F}_{q^2}.$$

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The projective general linear group $PGL(2, q)$ acts transitively on H_q by the fractional linear transformation:

$$g \cdot z = \frac{az + b}{cz + d},$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, q)$ and $z \in H_q$. Group theoretically, H_q is the homogeneous space $PGL(2, q)/Z_{q+1}$, where Z_{q+1} is a cyclic subgroup of order $q + 1$, and this is identical to $GL(2, q)/GL(1, q^2)$. Also note that the pair $(PGL(2, q), Z_{q+1})$ is a Gelfand-pair, namely, the permutation character of $PGL(2, q)$ acting on $PGL(2, q)/Z_{q+1}$ is multiplicity-free, or equivalently, the associated association scheme is commutative (we refer the reader to [1, 2] for the background in the theory of commutative association schemes). In fact, this particular association scheme satisfies the stronger condition that it is symmetric. For odd q, the character table P_1 of the association scheme corresponding to $PGL(2, q)/Z_{q+1}$ is described as follows:

$$P_1 = \left[\begin{array}{cc|ccc} 1 & 1 & q+1 & \dots & q+1 \\ \hline 1 & 1 & & & \\ \vdots & \vdots & & & \\ 1 & 1 & & (\psi_{ij})_{\substack{1 \le i \le q-1 \\ 1 \le j \le q-2}} & \\ 1 & -1 & & & \\ 1 & -1 & & & \\ \vdots & \vdots & & & \\ 1 & -1 & & & \end{array}\right]. \tag{1}$$

The entries ψ_{ij} are elements of $\mathbb{Q}(\zeta_{q-1}) \cup \mathbb{Q}(\zeta_{q+1})$, where $\zeta_n = \exp(2\pi\sqrt{-1}/n)$, and it is known that they are expressed by using power sums and Soto-Andrade sums (see [19, Chapters 19–21]).

A regular graph of valency k is called *Ramanujan* if all eigenvalues θ such that $|\theta| \neq k$ satisfy

$$|\theta| \leq 2\sqrt{k - 1}. \tag{2}$$

By combining works of A. Weil [20], P. Deligne, R. Evans [9, 10], H. Stark, N. Katz [11, 12], W. Li [14] and many others, it was proved that the finite upper half plane graphs of valency $q + 1$ over the finite field $\mathbb{F}_q$ are all Ramanujan graphs, that is, we have

$$|\psi_{ij}| \leq 2\sqrt{q}, \tag{3}$$

for all $1 \leq i \leq q - 1$, $1 \leq j \leq q - 2$.

Here, we give an example of an interesting family of multiplicity-free permutation groups, or commutative association schemes which are close relatives of the finite upper half planes over finite fields. Namely, we consider the homogeneous space $PGL(2, q)/Z_{q-1}$, where Z_{q-1} is a cyclic subgroup of order $q - 1$. Strictly speaking, it is known that the permutation group $PGL(2, q)$ on $PGL(2, q)/Z_{q-1}$ is not multiplicity-free (hence the associated association scheme is not commutative), but if we take

$G = Z_2 \times PGL(2, q) \cong GO_3(q)$ and $K = Z_2 \times D_{q-1} (\cong D_{2(q-1)})$ where D_{q-1} is a dihedral subgroup of order $q - 1$, then the permutation group G on G/K is multiplicity-free, and moreover, the associated association scheme is also symmetric. The association scheme G/K is of class $q + 1$, while the association scheme $PGL(2, q)/Z_{q+1}$ is of class $q - 1$. The character table P_2 of the association scheme G/K is of the following form (see [7, 13]):

$$P_2 = \begin{bmatrix} 1 & 1 & 2(q-1) & 2(q-1) & q-1 & \dots & q-1 \\ 1 & 1 & q-3 & q-3 & -2 & \dots & -2 \\ 1 & -1 & q-1 & -(q-1) & 0 & \dots & 0 \\ 1 & 1 & -2 & -2 & & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ 1 & 1 & -2 & -2 & & (-\psi_{ij})_{\substack{1 \le i \le q-1 \\ 1 \le j \le q-2}} & \\ 1 & -1 & -2 & 2 & & & \\ 1 & -1 & -2 & 2 & & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ 1 & -1 & -2 & 2 & & & \end{bmatrix}. \tag{4}$$

Note that the same quantities ψ_{ij} appear in both character tables. This is not an accident, but is a phenomenon called Ennola type duality, and was observed and explained in Bannai et al. [7]. In general, the graphs of valency $q - 1$ attached to G/K, i.e., the graphs of valency $q - 1$ whose edge sets are orbits of G on $G/K \times G/K$, are not Ramanujan (all the eigenvalues of these graphs appear in the corresponding columns of P_2). However, it is easy to see that if the condition $|\psi_{ij}| \le 2\sqrt{q-2}$ $(1 \le i \le q - 1)$ is satisfied for a fixed $j \in \{1, 2, \dots, q - 2\}$, then the graph of valency $q - 1$ corresponding to the column of P_2 is Ramanujan. As we will see in Section 8, our computer experiments show that for odd primes $p < 500$, this condition is satisfied for approximately 90 percent of cases of j. Therefore, we obtain many Ramanujan graphs with valency $p - 1$. Later, for $q = p^r$ with r odd, we will prove a rigorous (but easy) result $|\psi_{ij}| < 2\sqrt{q}$, which is weaker than $|\psi_{ij}| \le 2\sqrt{q-2}$, but better than the previous bound (3) (although we have assumed (3), see Lemma 2.1). We will use this result to obtain many other examples of families of Ramanujan graphs. See Theorems 2.2, 4.2 and 6.1.

The remaining part of the paper is devoted to showing that essentially the same methods work for far wider classes of association schemes. Namely, we consider all the association schemes considered in papers [5, 6, 17], and obtain many examples of families of Ramanujan graphs. Note that a similar problem was treated for finite analogues of Euclidean spaces in the companion paper [8] by the same authors.

Note that, by our constructions, we have only finitely many such graphs for each fixed valency, and so this does not give the answer whether we can construct infinitely many Ramanujan graphs with a fixed valency (cf. [15]). However, we still believe that the constructions of many examples of Ramanujan graphs given in this paper are interesting even if the valency is not fixed. In passing, as an immediate consequence of Bannai and Ito [4] on the spectra of distance-regular graphs, we will show that for each $k \ge 3$, there are only finitely many Ramanujan distance-regular graphs with valency k.

1. Graphs obtained from the action of $O_{2m+1}(q)$ (q: odd) on the set of non-square-type non-isotropic points

Let $V = V_{2m+1}(q)$ be the $(2m+1)$-dimensional vector space over the finite field $\mathbb{F}_q$ (q: odd), and $Q : V \longrightarrow \mathbb{F}_q$ the non-degenerate quadratic form on V with Witt index m:

$$Q(x) = 2(x_1 x_{m+1} + x_2 x_{m+2} + \cdots + x_m x_{2m}) + x_{2m+1}^2,$$

for $x = (x_1, x_2, \ldots, x_{2m+1}) \in V$. For each element x of V, we denote the 1-dimensional subspace containing x by $[x]$. Let Θ be the set of all non-square-type non-isotropic 1-dimensional subspaces of V, then we have $|\Theta| = \frac{1}{2}q^m(q^m - 1)$.

The simple orthogonal group $O_{2m+1}(q)$ acts transitively on Θ, and we denote the symmetric association scheme of class $(q+1)/2$ obtained from this action by $\mathfrak{X}(O_{2m+1}(q), \Theta)$. The relations of $\mathfrak{X}(O_{2m+1}(q), \Theta)$ are given as follows ([5, Section 7]):

$$([x],[y]) \in R_1 \;\Leftrightarrow\; \begin{pmatrix} x \\ y \end{pmatrix} \cdot S \cdot \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \nu & 1 \\ 1 & \nu^{-1} \end{pmatrix},$$

$$([x],[y]) \in R_i \;\Leftrightarrow\; \begin{pmatrix} x \\ y \end{pmatrix} \cdot S \cdot \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \nu & 1 \\ 1 & \nu^{2i-3} \end{pmatrix}, \qquad \left(2 \le i \le \frac{q-1}{2}\right)$$

$$([x],[y]) \in R_{(q+1)/2} \;\Leftrightarrow\; \begin{pmatrix} x \\ y \end{pmatrix} \cdot S \cdot \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix},$$

where $\nu \in \mathbb{F}_q$ is a primitive element of $\mathbb{F}_q$, A' denotes the transpose of a matrix A, and S is the matrix of the associated bilinear form of Q:

$$S = \begin{pmatrix} & I_m & \\ I_m & & \\ & & 1 \end{pmatrix}.$$

Note that $\mathfrak{X}(O_3(q), \Theta)$ is isomorphic to the association scheme $PGL(2,q)/D_{2(q+1)}$, where $D_{2(q+1)}$ is a dihedral subgroup of order $2(q+1)$, which is a quotient association scheme of $PGL(2,q)/Z_{q+1}$ (see [2, Section 2.9]). In particular, the character table $\tilde{P}^-$ of $\mathfrak{X}(O_3(q), \Theta)$ is described as

$$\tilde{P}^- = \begin{bmatrix} 1 & q+1 & \ldots & q+1 & \frac{1}{2}(q+1) \\ 1 & & & & \\ \vdots & & (\chi_{ij})_{\substack{1\le i\le (q-1)/2 \\ 1\le j\le (q-1)/2}} & & \\ 1 & & & & \end{bmatrix}. \tag{5}$$

Up to a suitable permutation of the columns of the matrix P_1 in the introduction, the entries χ_{ij} are given by

$$\chi_{ij} = \tfrac{1}{2}(\psi_{i,2j-1} + \psi_{i,2j}) \tag{6}$$

for $1 \le i \le (q-1)/2$ and $1 \le j \le (q-3)/2$, and

$$\chi_{i,(q-1)/2} = \tfrac{1}{2}\psi_{i,q-2} \tag{7}$$

for $1 \le i \le (q-1)/2$.

It is known that if $m > 1$ then the character table P^- of $\mathfrak{X}(O_{2m+1}(q), \Theta)$ is controlled by that of $\mathfrak{X}(O_3(q), \Theta)$. More precisely, it is expressed as follows:

$$P^- = \begin{bmatrix} 1 & (q^{m-1}-1)(q^m+1) & q^{m-1}(q^m+1) & \dots & q^{m-1}(q^m+1) & \frac{1}{2}q^{m-1}(q^m+1) \\ 1 & -(q-2)q^{m-1}-1 & 2q^{m-1} & \dots & 2q^{m-1} & q^{m-1} \\ 1 & q^{m-1}-1 & & & & \\ \vdots & \vdots & & (q^{m-1}\chi_{ij})_{\substack{1\le i\le (q-1)/2 \\ 1\le j\le (q-1)/2}} & & \\ 1 & q^{m-1}-1 & & & & \end{bmatrix}. \tag{8}$$

Therefore, by (3), (6) and (7) we have

Theorem 1.1. *The graphs* (Θ, R_i) $(2 \le i \le (q-1)/2)$ *of valency* $q^{m-1}(q^m+1)$ *and the graph* $(\Theta, R_{(q+1)/2})$ *of valency* $\frac{1}{2}q^{m-1}(q^m+1)$ *are always Ramanujan. The graph* (Θ, R_1) *is Ramanujan if and only if* $q = 3, 5$*, or* $q = 7$ *and* $m \ge 3$.

2. Graphs obtained from the action of $O_{2m+1}(q)$ (q: odd) on the set of square-type non-isotropic points

We keep the notation in the previous section. Let Ω be the set of square-type non-isotropic 1-dimensional subspaces of $V = V_{2m+1}(q)$ $(|\Omega| = \frac{1}{2}q^m(q^m+1))$. Then, $O_{2m+1}(q)$ acts transitively on Ω, and yields a symmetric association scheme $\mathfrak{X}(O_{2m+1}(q), \Omega)$ of class $(q+1)/2$. The relations of $\mathfrak{X}(O_{2m+1}(q), \Omega)$ are given by

$$R_1 = \{([x],[y]) \in \Omega \times \Omega \mid Q(x+y) = 0\},$$

$$R_i = \{([x],[y]) \in \Omega \times \Omega \mid Q(x+y) = 2 + 2\nu^{-(i-1)}\}, \qquad \left(2 \le i \le \frac{q-1}{2}\right)$$

$$R_{(q+1)/2} = \{([x],[y]) \in \Omega \times \Omega \mid Q(x+y) = 2\},$$

where we assume $Q(x) = 1$ for all $[x] \in \Omega$ ([5, Section 6]).

Note that $\mathfrak{X}(O_3(q), \Omega)$ is isomorphic to the association scheme $PGL(2,q)/D_{2(q-1)}$, where $D_{2(q-1)}$ is a dihedral subgroup of order $2(q-1)$. The character table $\tilde{P}^+$ of $\mathfrak{X}(O_3(q), \Omega)$ is closely related to that of $\mathfrak{X}(O_3(q), \Theta)$. Namely, we have

$$\tilde{P}^+ = \begin{bmatrix} 1 & 2(q-1) & q-1 & \dots & q-1 & \frac{1}{2}(q-1) \\ 1 & q-3 & -2 & \dots & -2 & -1 \\ 1 & -2 & & & & \\ \vdots & \vdots & & (-\chi_{ij})_{\substack{1\le i\le (q-1)/2 \\ 1\le j\le (q-1)/2}} & & \\ 1 & -2 & & & & \end{bmatrix}. \tag{9}$$

This is another example of Ennola type duality observed in [7].

The character table P^+ of $\mathfrak{X}(O_{2m+1}(q), \Omega)$ is controlled by that of $\mathfrak{X}(O_3(q), \Omega)$:

$$P^+ = \begin{bmatrix} 1 & (q^{m-1}+1)(q^m-1) & q^{m-1}(q^m-1) & \dots & q^{m-1}(q^m-1) & \frac{1}{2}q^{m-1}(q^m-1) \\ 1 & (q-2)q^{m-1}-1 & -2q^{m-1} & \dots & -2q^{m-1} & -q^{m-1} \\ 1 & -(q^{m-1}+1) & & & & \\ \vdots & \vdots & & (-q^{m-1}\chi_{ij})_{\substack{1\le i\le (q-1)/2 \\ 1\le j\le (q-1)/2}} & & \\ 1 & -(q^{m-1}+1) & & & & \end{bmatrix}. \tag{10}$$

The graphs (Ω, R_i) are not Ramanujan in general, but if $q = p^r$ with r odd, then we can still obtain many Ramanujan graphs from this association scheme. To see this, we need the following lemma which is a slight strengthening of (3).

Lemma 2.1. *If $q = p^r$ with r odd, then we have*

$$|\psi_{ij}| < 2\sqrt{q},$$

for all $1 \le i \le q-1$, $1 \le j \le q-2$.

Proof. By virtue of (3), we only have to show $|\psi_{ij}| \ne 2\sqrt{q}$. Now, let

$$N = \begin{cases} p & \text{if } p \equiv 1 \pmod 4 \\ 4p & \text{if } p \equiv 3 \pmod 4). \end{cases}$$

Then, it is well known that $\mathbb{Q}(\sqrt{p}) \subset \mathbb{Q}(\zeta_N)$, where $\zeta_n = \exp(2\pi\sqrt{-1}/n)$, and that N is the smallest natural number which satisfies this property. Since $\mathbb{Q}(\zeta_N) \cap \mathbb{Q}(\zeta_{q\pm1}) = \mathbb{Q}(\zeta_{(N,q\pm1)})$ and $(N, q\pm1) < N$, we conclude $\sqrt{p} \notin \mathbb{Q}(\zeta_{q\pm1})$. Therefore, since $|\psi_{ij}|$ is contained in $\mathbb{Q}(\zeta_{q+1}) \cup \mathbb{Q}(\zeta_{q-1})$, it cannot be equal to $2\sqrt{q}$. □

Theorem 2.2. *We have*

(i) *If $q = p^r$ with r odd, then the graphs (Ω, R_i) $(2 \le i \le (q-1)/2)$ of valency $q^{m-1}(q^m-1)$ are Ramanujan if m is sufficiently large (i.e., larger than a certain number which is determined by q).*

(ii) *The graph $(\Omega, R_{(q+1)/2})$ of valency $\frac{1}{2}q^{m-1}(q^m-1)$ is Ramanujan if $m \ge 2$, or if $m = 1$ and $q \ge 7$.*

(iii) *The graph (Ω, R_1) is Ramanujan if and only if $q = 3, 5, 7$, or $q = 9, 11$ and $m = 1$.*

Proof. We prove (i). The graphs (Ω, R_i) $(2 \le i \le (q-1)/2)$ are regular with valency $k = q^{m-1}(q^m-1)$. Let $|\chi_{ij}| = C_{ij}\sqrt{q}$, then it follows from Lemma 2.1 that $0 \le C_{ij} < 2$. Therefore,

$$(2\sqrt{k-1})^2 - |q^{m-1}\chi_{ij}|^2 = (4 - C_{ij}^2)q^{2m-1} - 4q^{m-1} - 4$$

is positive when m is large, so that satisfies (2).

The proof of (ii) and (iii) are straightforward, hence omitted. □

Remark 2.1. Note that if $|\chi_{ij}| \le 2\sqrt{q-2}$ $(1 \le i \le (q-1)/2)$ for a fixed $j \in \{1, 2, \dots, (q-1)/2\}$, then the graph (Ω, R_j) is a Ramanujan graph. In the last section, we give computer experiments on this condition.

3. Graphs obtained from the action of $O_{2m+1}(q)$ (q: even) on the set of negative-type hyperplanes

In Sections 3 and 4, we consider the action of the orthogonal group $O_{2m+1}(q)$ (q: even) on the set of negative-type hyperplanes and on that of positive-type hyperplanes, respectively (note that the action of $O_{2m+1}(q)$ on non-isotropic projective points is not transitive).

Let $V = V_{2m+1}(q)$ be the $(2m+1)$-dimensional vector space over the finite field $\mathbb{F}_q$ (q: even), and $Q : V \longrightarrow \mathbb{F}_q$ the non-degenerate quadratic form on V with Witt index m:

$$Q(x) = x_1x_{m+1} + x_2x_{m+2} + \cdots + x_mx_{2m} + x_{2m+1}^2,$$

for $x = (x_1, x_2, \ldots, x_{2m+1}) \in V$. Let Θ be the set of positive-type hyperplanes of V, then we have $|\Theta| = \frac{1}{2}q^m(q^m - 1)$ and the orthogonal group $O_{2m+1}(q)$ acts transitively on Θ. We denote the association scheme obtained from this action by $\mathfrak{X}(O_{2m+1}(q), \Theta)$. We describe the relations of $\mathfrak{X}(O_{2m+1}(q), \Theta)$ following [17].

Let U, U' be elements of Θ, and suppose that $U \cap U'$ is non-degenerate. Then, there exists a vector w of $U \cap U'$ such that $Q(w) = 1$ and

$$U \cap U' = \langle w\rangle^\perp \cap U = \langle w\rangle^\perp \cap U',$$

where $\langle w\rangle^\perp$ is the orthogonal complement of $\langle w\rangle$ with respect to the associated bilinear form $f : V \times V \longrightarrow \mathbb{F}_q$ of Q. It is not very difficult to show that for any non-degenerate hyperplane W of $U \cap U'$, there exist two vectors $u \in U$ and $u' \in U'$ such that $Q(u) = Q(u')$, $f(u, w) = f(u', w) = 1$ and

$$U = \langle u, w\rangle \perp W, \qquad U' = \langle u', w\rangle \perp W.$$

Now, we define

$$\Delta = \Delta(W; u, u') = \frac{f(u, u')}{f(u, u') + 1}.$$

Then, it is shown that $\Delta \in \mathbb{F}_q\backslash\{0, 1\}$ and the pair $\{\Delta, \Delta^{-1}\}$ does not depend on the choice of W, u and u'. Put

$$\Delta(U, U') = \{\Delta, \Delta^{-1}\}.$$

Then, the relations of $\mathfrak{X}(O_{2m+1}(q), \Theta)$ are given as follows.

$$\begin{aligned} R_1 &= \{(U, U') \in \Theta \times \Theta \mid U \cap U' : \text{ degenerate}\},\\ R_i &= \{(U, U') \in \Theta \times \Theta \mid U \cap U' : \text{ non-degenerate},\\ &\qquad \Delta(U, U') = \{\nu^{i-1}, \nu^{-(i-1)}\}\}, \qquad \left(2 \le i \le \frac{q}{2}\right) \end{aligned}$$

where, as usual, ν is a primitive element of $\mathbb{F}_q$. In particular, $\mathfrak{X}(O_{2m+1}, \Theta)$ is a symmetric association scheme of class $q/2$ if $m \ge 2$.

Since $\mathfrak{X}(O_3(q), \Theta)$ is isomorphic to the association scheme $PGL(2, q)/D_{2(q+1)}$, where $D_{2(q+1)}$ is a dihedral subgroup of order $2(q+1)$, it is a quotient association scheme of $GL(2, q)/GL(1, q^2)$. The character table $\tilde{P}^-$ of $\mathfrak{X}(O_3(q), \Theta)$ is described as follows:

$$\tilde{P}^- = \begin{bmatrix} 1 & q+1 & \dots & q+1 \\ 1 & & & \\ \vdots & & (\phi_{ij})_{\substack{1\le i\le (q-2)/2 \\ 1\le j\le (q-2)/2}} & \\ 1 & & & \end{bmatrix}. \tag{11}$$

The values $\phi_{ij} \in \mathbb{Q}(\zeta_{q-1})$ are explicitly calculated in [18] (see (14)).

Evans [10] defined the finite upper half planes over the finite fields $\mathbb{F}_q$ of characteristic 2 which are identical to the homogeneous spaces $GL(2, q)/GL(1, q^2)$, and showed that the finite upper half plane graphs of valency $q+1$ are Ramanujan for even q as well, using a result of Katz [12]. This implies

$$|\phi_{ij}| \le 2\sqrt{q}, \tag{12}$$

for all $1 \le i \le (q-2)/2$ and $1 \le j \le (q-2)/2$.

For $m \ge 2$, the character table P^- of $\mathfrak{X}(O_{2m+1}(q), \Theta)$ is described as

$$P^- = \begin{bmatrix} 1 & (q^{m-1}-1)(q^m+1) & q^{m-1}(q^m+1) & \dots & q^{m-1}(q^m+1) \\ 1 & -(q-2)q^{m-1}-1 & 2q^{m-1} & \dots & 2q^{m-1} \\ 1 & q^{m-1}-1 & & & \\ \vdots & \vdots & & (q^{m-1}\phi_{ij})_{\substack{1\le i\le (q-2)/2 \\ 1\le j\le (q-2)/2}} & \\ 1 & q^{m-1}-1 & & & \end{bmatrix}. \tag{13}$$

From this table and (12), we have the following:

Theorem 3.1. *The graphs (Θ, R_i) $(2 \le i \le q/2)$ of valency $q^{m-1}(q^m+1)$ are always Ramanujan. The graph (Θ, R_1) is Ramanujan if and only if $q = 2, 4$.*

4. Graphs obtained from the action of $O_{2m+1}(q)$ (q: even) on the set of positive-type hyperplanes

Assume that q is even. Then, the transitive action of the orthogonal group $O_{2m+1}(q)$ on the set Ω of positive-type hyperplanes of $V = V_{2m+1}(q)$, with respect to the quadratic form $Q(x) = x_1x_{m+1} + \cdots + x_mx_{2m} + x_{2m+1}^2$, defines a symmetric association scheme $\mathfrak{X}(O_{2m+1}(q), \Omega)$ of class $q/2$ ($|\Omega| = \frac{1}{2}q^m(q^m+1)$). We define the "angle" $\Delta(U, U')$ between two elements U, U' of Ω such that $U \cap U'$ is non-degenerate, in exactly the same manner as in the previous section. Then, the association classes of $\mathfrak{X}(O_{2m+1}(q), \Omega)$ are given by

$$\begin{aligned} R_1 &= \{(U, U') \in \Omega \times \Omega \mid U \cap U' : \text{ degenerate}\}, \\ R_i &= \{(U, U') \in \Omega \times \Omega \mid U \cap U' : \text{ non-degenerate}, \\ &\quad \Delta(U, U') = \{\nu^{i-1}, \nu^{-(i-1)}\}\}, \qquad \left(2 \le i \le \frac{q}{2}\right). \end{aligned}$$

Clearly, $\mathfrak{X}(O_3(q), \Omega)$ is isomorphic to the association scheme $PGL(2,q)/D_{2(q-1)}$, where $D_{2(q-1)}$ is a dihedral subgroup, and the character table $\tilde{P}^+$ of $PGL(2,q)/D_{2(q-1)}$ is calculated explicitly in [18]. We can observe Ennola type duality also in this case. Namely,

$$\tilde{P}^+ = \begin{bmatrix} 1 & 2(q-1) & q-1 & \dots & q-1 \\ 1 & q-3 & -2 & \dots & -2 \\ 1 & -2 & & & \\ \vdots & \vdots & & (-\phi_{ij})_{\substack{1\le i\le (q-2)/2 \\ 1\le j\le (q-2)/2}} & \\ 1 & -2 & & & \end{bmatrix}. \tag{14}$$

The character table P^+ of $\mathfrak{X}(O_{2m+1}(q), \Omega_{2m+1}(q))$ is controlled by $\tilde{P}^+$:

$$P^+ = \begin{bmatrix} 1 & (q^{m-1}+1)(q^m-1) & q^{m-1}(q^m-1) & \dots & q^{m-1}(q^m-1) \\ 1 & (q-2)q^{m-1}-1 & -2q^{m-1} & \dots & -2q^{m-1} \\ 1 & -(q^{m-1}+1) & & & \\ \vdots & \vdots & & (-q^{m-1}\phi_{ij})_{\substack{1\le i\le (q-2)/2 \\ 1\le j\le (q-2)/2}} & \\ 1 & -(q^{m-1}+1) & & & \end{bmatrix}. \tag{15}$$

Lemma 4.1. *If $q = 2^r$ with r odd, then we have*

$$|\phi_{ij}| < 2\sqrt{q},$$

for all $1 \le i \le (q-2)/2$, $1 \le j \le (q-2)/2$.

Proof. The proof is similar to that of Lemma 2.1, hence we omit the detail. □

Finally, we obtain the following theorem.

Theorem 4.2. *We have*

(i) *If $q = 2^r$ with r odd, then the graphs (Ω, R_i) $(2 \le i \le q/2)$ of valency $q^{m-1}(q^m-1)$ are Ramanujan if m is sufficiently large (i.e., larger than a certain number which is determined by q).*
(ii) *The graph (Ω, R_1) is Ramanujan if and only if $q = 2, 4$, or $q = 8$ and $m = 1, 2$.*

5. Graphs obtained from the action of $O^{\pm}_{2m}(q)$ on the set of non-isotropic points

In this section, we will see that we can obtain many Ramanujan graphs from the action of the finite (simple) orthogonal groups $O^{\pm}_{2m}(q)$ on the set of non-isotropic (projective) points, which is considered in [5, Sections 2–5]. In order to keep this paper concise, we only deal with the action of $O^{+}_{2m}(q)$ with q even, but the other cases are similar.

Assume that q is even. Let $V = V_{2m}(q)$ be the $2m$-dimensional vector space over $\mathbb{F}_q$, and $Q^+ : V \longrightarrow \mathbb{F}_q$ be the non-degenerate quadratic form on V with Witt index m:

$$Q^+(x) = x_1x_{m+1} + x_2x_{m+2} + \dots + x_mx_{2m},$$

for $x = (x_1, x_2, \dots, x_{2m}) \in V$. As in Section 1, for each element x of V we denote the projective point containing x by $[x]$. Let Ω be the set of non-isotropic projective points of V. Note that since q is even, for each element of Ω we can always take a unique representative $x \in V$ such that $Q^+(x) = 1$, therefore we may identify Ω with the set $\{x \in V \mid Q^+(x) = 1\}$.

The orthogonal group $O^+_{2m}(q)$ acts transitively on Ω, and we denote the association scheme obtained from this action by $\mathfrak{X}(O^+_{2m}(q), \Omega)$. With the above identification, the relations of $\mathfrak{X}(O^+_{2m}(q), \Omega)$ are given as follows ([5, Section 2]):

$$R_1 = \{(x, y) \in \Omega \times \Omega \mid x \neq y, Q^+(x + y) = 0\},$$
$$R_i = \{(x, y) \in \Omega \times \Omega \mid Q^+(x + y) = \nu^{i-1}\}, \qquad (2 \leq i \leq q)$$

where $\nu \in \mathbb{F}_q$ is a primitive element of $\mathbb{F}_q$. In particular, $\mathfrak{X}(O^+_{2m}(q), \Omega)$ is a symmetric association scheme of class q. The valencies k_i of the regular graphs (Ω, R_i) are given by

$$k_1 = (q^{m-1} + 1)(q^{m-1} - 1),$$
$$k_i = \begin{cases} q^{m-1}(q^{m-1} - 1) & \text{if } t^2 + \nu^{i-1}t + 1 : \text{ irreducible over } \mathbb{F}_q, \\ q^{m-1}(q^{m-1} + 1) & \text{if } t^2 + \nu^{i-1}t + 1 : \text{ reducible over } \mathbb{F}_q, \end{cases}$$

for $2 \leq i \leq q$.

It is known that the character table P of $\mathfrak{X}(O^+_{2m}(q), \Omega)$ is controlled by that of the group association scheme of the special linear group $SL(2, q)$, essentially by the replacement $q \longmapsto q^{m-1}$. More precisely, up to a suitable permutation of the columns, we have

$$P = \left[\begin{array}{ccccccccc} 1 & (q^{m-1}+1)(q^{m-1}-1) & q^{m-1}(q^{m-1}-1) & \dots & q^{m-1}(q^{m-1}-1) & q^{m-1}(q^{m-1}+1) & \dots & q^{m-1}(q^{m-1}+1) \\ 1 & q^{m-2}-1 & -q^{m-2}(q-1) & \dots & -q^{m-2}(q-1) & q^{m-2}(q+1) & \dots & q^{m-2}(q+1) \\ 1 & -(q^{m-1}+1) & & & & & & \\ \vdots & \vdots & & (-q^{m-1}(\sigma^{ij}+\sigma^{-ij}))_{\substack{1\leq i\leq q/2 \\ 1\leq j\leq q/2}} & & & 0 & \\ 1 & -(q^{m-1}+1) & & & & & & \\ 1 & q^{m-1}-1 & & & & & & \\ \vdots & \vdots & & 0 & & & (q^{m-1}(\rho^{kl}+\rho^{-kl}))_{\substack{1\leq k\leq (q-2)/2 \\ 1\leq l\leq (q-2)/2}} & \\ 1 & q^{m-1}-1 & & & & & & \end{array}\right] \tag{16}$$

where $\sigma = \exp(2\pi\sqrt{-1}/(q + 1))$ and $\rho = \exp(2\pi\sqrt{-1}/(q - 1))$. Note that unless $q + 1$ is prime, some of the entries $-q^{m-1}(\sigma^{ij} + \sigma^{-ij})$ attain the absolute value $2q^{m-1}$.

From this table, we immediately obtain the following:

Theorem 5.1. (i) *If $t^2 + \nu^{i-1}t + 1$ is reducible over $\mathbb{F}_q$, then the graph (Ω, R_i) of valency $q^{m-1}(q^{m-1}+1)$ is a Ramanujan graph.*

(ii) *Suppose that $q+1$ is prime and that $t^2 + \nu^{i-1}t + 1$ is irreducible over $\mathbb{F}_q$. Then, the graph (Ω, R_i) of valency $q^{m-1}(q^{m-1}-1)$ is Ramanujan if m is sufficiently large (i.e., larger than a certain number which is determined by q).*

(iii) *The graph (Ω, R_1) is Ramanujan unless $(q, m) = (2, 2)$.*

6. Graphs obtained from the action of $U_n(q)$ on the set of non-isotropic points

In this section, we consider the action of the simple unitary group $U_n(q)$ on the set of non-isotropic projective points.

Let $V = V_n(q^2)$ be the n-dimensional vector space over $\mathbb{F}_{q^2}$ ($n \geq 3$), and $H : V \longrightarrow \mathbb{F}_q$ the canonical non-singular Hermitian form:

$$H(x) = x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n,$$

for $(x_1, x_2, \ldots, x_n) \in V$. We denote the 1-dimensional subspace containing $x \in V$ by $[x]$. Let Ω be the set of 1-dimensional non-isotropic subspaces of V with respect to H. Then, we have

$$|\Omega| = \frac{q^{n-1}(q^n - (-1)^n)}{q+1},$$

and $U_n(q)$ acts transitively on Ω. We denote the association scheme obtained from this action by $\mathfrak{X}(U_n(q), \Omega)$. The relations of $\mathfrak{X}(U_n(q), \Omega)$ are described as follows ([6, Section 1]).

$$([x],[y]) \in R_1 \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \cdot \overline{\begin{pmatrix} x \\ y \end{pmatrix}}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$([x],[y]) \in R_i \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \cdot \overline{\begin{pmatrix} x \\ y \end{pmatrix}}' = \begin{pmatrix} 1 & 1 \\ 1 & \nu^{i-1} \end{pmatrix}, \qquad (2 \leq i \leq q)$$

where ν is a primitive element of $\mathbb{F}_q$. In particular, $\mathfrak{X}(U_n(q), \Omega)$ ($n \geq 3$) is a symmetric association scheme of class q.

The character table P of $\mathfrak{X}(U_n(q), \Omega)$ ($n \geq 3$) is controlled by that of $PGL(2, q)/Z_{q+1}$:

$$P = \begin{bmatrix} 1 & \frac{q^{n-2}(q^{n-1}+(-1)^n)}{q+1} & q^{n-2}(q^{n-1}+(-1)^n) & \ldots & q^{n-2}(q^{n-1}+(-1)^n) \\ 1 & (-1)^n q^{n-2} & & & \\ \vdots & \vdots & & & \\ 1 & (-1)^n q^{n-2} & & ((-1)^n q^{n-2}\psi_{ij})_{\substack{1\leq i\leq q-1 \\ 1\leq j\leq q-2}} & \\ 1 & -(-1)^n q^{n-2} & & & \\ \vdots & \vdots & & & \\ 1 & -(-1)^n q^{n-2} & & & \\ 1 & (-1)^n q^{n-3} & (-1)^n q^{n-3}(q+1) & \ldots & (-1)^n q^{n-3}(q+1) \end{bmatrix}$$

$$\left.\begin{matrix} q^{2n-3}-(-1)^n(q-1)q^{n-2}-1 \\ (-1)^n q^{n-2}-1 \\ \vdots \\ (-1)^n q^{n-2}-1 \\ (-1)^n q^{n-2}-1 \\ \vdots \\ (-1)^n q^{n-2}-1 \\ (-1)^{n-1}q^{n-3}(q^2-q-1)-1 \end{matrix}\right] \tag{17}$$

from which it follows that

Theorem 6.1. (i) *If $n = 2m$ is even, then the graphs (Ω, R_i) $(2 \le i \le q-1)$ of valency $q^{n-2}(q^{n-1}+1)$ are all Ramanujan.*

(ii) *If $n = 2m+1$ is odd, then the graphs (Ω, R_i) $(2 \le i \le q-1)$ of valency $q^{n-2}(q^{n-1}-1)$ are Ramanujan if m is sufficiently large (i.e., larger than a certain number which is determined by q).*

(ii) *The graph (Ω, R_1) is always Ramanujan.*

(iii) *The graph (Ω, R_q) is Ramanujan if and only if $q \le 5$.*

7. Concluding remarks

Remark 7.1. Bannai et al. [6, Section 2] showed that the character table P_2 in the introduction controls that of the association scheme defined by the action of the symplectic group $Sp_{2m}(q)$ on the set of non-isotropic lines. However, when $m \ge 3$, we generally do not obtain Ramanujan graphs from this action, and therefore, we did not treat this case here. Please note that for $m = 2$ we still obtain many Ramanujan graphs.

Remark 7.2. We can show that for each fixed $k \ge 3$ there are only finitely many distance-regular Ramanujan graphs, by the argument used in the series of papers [4]. In fact, by using the interlace property of the eigenvalues, it can be proved that if a distance-regular graph of valency k is Ramanujan, then the intersection array must have the following form:

$$\left\{\begin{matrix} 0 & 1 & \cdots & 1 & c_{s+1} & \cdots & c_{s+t} & k-1 & \cdots & k-1 & c_d \\ k & 0 & \cdots & 0 & a_{s+1} & \cdots & a_{s+t} & 0 & \cdots & 0 & a_d \\ * & k-1 & \cdots & k-1 & b_{s+1} & \cdots & b_{s+t} & 1 & \cdots & 1 & * \end{matrix}\right\},$$

where $(c_s, a_s, b_s) \ne (1, 0, k-1)$, $(c_{s+t}, a_{s+t}, b_{s+t}) \ne (k-1, 0, 1)$ and t is bounded by a certain constant which is determined by k. Then, Theorem 1 in [4] implies that there are only finitely many distance-regular graphs with such intersection array.

Also, it is interesting to remark that the Ramanujan bound $2\sqrt{k-1}$ is also relevant in the study of the spectra of distance-regular graphs, see [3, 4].

8. Computer calculations

As mentioned in the Introduction and Remark 2.1, the condition

$$|\psi_{ij}| \le 2\sqrt{q-2} \qquad (1 \le i \le q-1) \tag{$*$}$$

Table 1

p	δ	a	α	β	$2\sqrt{p-2}$	Ans.
17	3	1	6.1061	7.2909	7.7460	true
17	3	11	6.3592	7.2909	7.7460	true
17	3	2	6.7169	6.0000	7.7460	true
17	3	10	6.0000	6.7169	7.7460	true
17	3	3	7.2263	6.0000	7.7460	true
17	3	9	6.0000	7.2263	7.7460	true
17	3	4	7.2893	6.0000	7.7460	true
17	3	8	6.0000	7.2893	7.7460	true
17	3	5	6.6405	7.0642	7.7460	true
17	3	7	6.8229	7.0642	7.7460	true
17	3	6	7.6569	8.0000	7.7460	false
17	3	13	6.4534	7.2893	7.7460	true
17	3	16	7.2893	6.0000	7.7460	true
17	3	14	7.6569	6.2490	7.7460	true
17	3	15	7.6569	5.1702	7.7460	true

for a fixed $j \in \{1, 2, \dots, q-2\}$ ensures the existence of families of Ramanujan graphs. In this section, we give the results of our computer experiments on this condition.

Let δ be a non-square element of $\mathbb{F}_q^\times$ (q: odd), and let H_q be the finite upper half plane as in the introduction. For $z = x + y\sqrt{\delta}$, $w = u + v\sqrt{\delta} \in H_q$, define the "distance" $d(z, w) \in \mathbb{F}_q$ by

$$d(z, w) = \frac{(x-u)^2 - \delta(y-v)^2}{yv}.$$

Then, the relations of the association scheme $PGL(2, q)/Z_{q+1}$ can be described as

$$R_a = \{(z, w) \in H_q \times H_q \mid d(z, w) = a\} \qquad (a \in \mathbb{F}_q^\times)$$

and the finite upper half plane graph $X_q(\delta, a) = (H_q, R_a)$ is of valency $q + 1$ if and only if $a \neq 4\delta$ ([19, Chapter 19]). The eigenvalues of these graphs are determined in [9, 16]. We evaluate the eigenvalues when $q = p$ are odd primes, by using the computer package "M A G M A" (http://magma.maths.usyd.edu.au/magma).

Table 1 is an example of our computer calculations.

In Table 1, $\alpha = \alpha_a$ and $\beta = \beta_a$ stand for the absolute value of the smallest eigenvalue and the second largest eigenvalue of the graph $X_{17}(3, a)$, respectively. The last column indicates whether the condition $(*)$ holds or not. All the other results (from $p = 3$ to 691) are available at the second author's homepage: http://www.math.kyushu-u.ac.jp/~ma299019/FUHP/.

Finally, we list in Table 2 the ratio of $a \in \mathbb{F}_p \backslash \{0, 4\delta\}$ satisfying the condition $(*)$, for all the odd primes $p < 500$.

Table 2

p	δ	$a \in \mathbb{F}_p \setminus \{0, 4\delta\}$ which do not satisfy (∗)	%
3	2		100
5	2	4	67
7	3	6	80
11	2	4	89
13	2	1, 4, 7	73
17	3	6	93
19	2	3, 4, 5	82
23	5	4, 6, 10, 14, 16	76
29	2	4, 7, 2, 6, 1	81
31	3	6, 16, 24, 19, 27	83
37	2	5, 23, 4, 22, 3	86
41	3	2, 6, 10, 25, 28	87
43	2	4, 10, 41	93
47	5	6, 14, 10, 23, 44	89
53	2	1, 7, 4, 10, 51, 27, 34	86
59	2	4, 32, 35	95
61	2	4, 17, 52, 18, 51, 20, 49, 22, 47, 27, 42	81
67	2	4, 23, 52	95
71	7	3, 25, 14, 45, 54	93
73	5	10, 26, 67, 27, 66, 30, 63, 41, 52	87
79	3	1, 11, 6, 14, 77, 22, 69, 39, 52	88
83	2	4, 10, 81, 17, 74, 23, 68, 25, 66, 31, 60, 38, 53	84
89	3	6, 23, 78	97
97	5	5, 15, 10, 34, 83, 41, 76, 45, 72, 47, 70, 48, 69	86
101	2	2, 6, 4, 9, 100, 20, 89, 21, 88, 22, 87, 39, 70, 51, 58	85
103	3	6, 15, 100, 19, 96, 32, 83, 38, 77, 40, 75, 45, 70, 56, 59, 57, 58	83
107	2	4, 28, 87, 31, 84, 37, 78, 54, 61	91
109	2	2, 6, 4, 15, 102, 17, 100, 25, 92, 35, 82, 38, 79	88
113	3	2, 10, 6, 20, 105, 22, 103, 23, 102, 30, 95, 32, 93, 49, 76	86
127	3	3, 9, 6, 25, 114, 38, 101, 48, 91, 60, 79	91
131	2	2, 6, 4, 40, 99, 43, 96, 62, 77, 65, 74, 68, 71	90
137	3	6, 15, 134, 17, 132, 24, 125, 37, 112, 44, 105, 54, 95, 73, 76, 74, 75	87
139	2	4, 13, 134, 14, 133, 54, 93, 63, 84, 64, 83, 69, 78, 72, 75	89
149	2	4, 23, 134, 41, 116, 43, 114, 54, 103, 55, 102, 58, 99, 59, 98, 60, 97, 61, 96, 62, 95	86
151	3	6, 27, 136, 42, 121, 47, 116, 49, 114, 78, 85	93
157	2	2, 6, 4, 18, 147, 31, 134, 33, 132, 40, 125, 48, 117, 50, 115, 51, 114, 61, 104, 66, 99, 72, 93	85
163	2	4, 36, 135, 39, 132, 42, 129, 58, 113, 60, 111, 82, 89	92
167	5	9, 11, 10, 22, 165, 26, 161, 28, 159, 68, 119, 9, 11, 10, 22, 165, 26, 161, 28, 159, 68, 119	87
173	2	1, 7, 4, 14, 167, 15, 166, 17, 164, 35, 146, 60, 121, 61, 120, 73, 108, 77, 104	89
179	2	4, 10, 177, 14, 173, 27, 160, 35, 152, 90, 97	94
181	2	4, 22, 167, 29, 160, 38, 151, 66, 123, 71, 118, 72, 117	93
191	7	14, 29, 190, 42, 177, 43, 176, 47, 172, 56, 163, 71, 148, 87, 132, 96, 123, 108, 111	90
193	5	2, 18, 5, 15, 10, 39, 174, 59, 154, 72, 141, 86, 127	93
197	2	4, 19, 186, 28, 177, 36, 169, 39, 166, 67, 138, 92, 113, 102, 103	92
199	3	6, 18, 193, 19, 192, 34, 177, 45, 166, 50, 161, 55, 156, 63, 148, 68, 143, 76, 135, 143, 76, 135, 77, 134	89
211	2	4, 31, 188, 76, 143, 83, 136	97
223	3	6, 18, 217, 43, 192, 78, 157, 105, 130	96

Table 2 (*continued*)

p	δ	$a \in \mathbb{F}_p \backslash \{0, 4\delta\}$ which do not satisfy (∗)	%
227	2	4, 42, 193, 78, 157, 81, 87, 148, 99, 136, 105, 130, 113, 122	94
229	2	4, 13, 224, 42, 195, 48, 189, 49, 188, 63, 174, 87, 150, 94, 143, 110, 127, 113, 124	92
233	3	6, 17, 228, 32, 213, 42, 203, 50, 195, 59, 186, 70, 175, 95, 150, 103, 142	93
239	7	14, 41, 226, 58, 209, 60, 207, 65, 202, 66, 201, 67, 200, 68, 199, 80, 187, 98, 169, 108, 159, 114, 153, 122, 145	89
241	7	8, 20, 14, 46, 223, 47, 222, 48, 221, 50, 219, 55, 214, 61, 208, 64, 205, 71, 198, 101, 168, 134, 135	90
251	2	4, 16, 243, 22, 237, 40, 219, 44, 215, 52, 207, 56, 203, 76, 183, 88, 171, 102, 157	92
257	3	6, 20, 249, 26, 243, 29, 240, 51, 218, 79, 190, 104, 165	95
263	5	4, 16, 7, 13, 8, 12, 10, 43, 240, 57, 226, 81, 202, 123, 160, 137, 146	93
269	2	4, 27, 250, 41, 236, 47, 230, 69, 208, 93, 184, 123, 154, 126, 151	94
271	3	6, 24, 259, 25, 258, 40, 243, 52, 231, 96, 187, 115, 168, 126, 157, 132, 151, 133, 150, 136, 147, 139, 144	91
277	2	4, 22, 263, 44, 241, 63, 222, 76, 209, 78, 207, 87, 198, 102, 183, 117, 168	94
281	3	6, 31, 262, 37, 256, 39, 254, 43, 250, 47, 246, 52, 241, 55, 238, 64, 229, 97, 196, 119, 174, 123, 170, 131, 162	91
283	2	4, 16, 275, 18, 273, 30, 261, 36, 255, 62, 229, 85, 206, 94, 197, 106, 185, 116, 175	93
293	2	4, 38, 263, 65, 236, 82, 219, 88, 213, 95, 206, 100, 201, 104, 197, 122, 179, 137, 164	93
307	2	4, 21, 294, 48, 267, 60, 255, 70, 245, 73, 242, 106, 209, 108, 207, 148, 167, 156, 159	94
311	11	22, 63, 292, 68, 287, 84, 271, 101, 254, 108, 247, 146, 209, 168, 187, 175, 180	94
313	5	6, 14, 10, 40, 293, 41, 292, 48, 285, 56, 277, 79, 254, 107, 226, 154, 179	95
317	2	4, 59, 266, 61, 264, 63, 262, 97, 228, 156, 169	97
331	2	4, 10, 329, 42, 297, 66, 273, 75, 264, 83, 256, 135, 204	96
337	5	10, 40, 317, 68, 289, 85, 272, 89, 268, 93, 264, 99, 258	96
347	2	4, 23, 332, 28, 327, 51, 304, 59, 296, 63, 292, 93, 262, 110, 245, 135, 220, 140, 215, 153, 202	94
349	2	4, 43, 314, 70, 287, 86, 271, 87, 270, 95, 262, 147, 210, 150, 207, 165, 192, 174, 183	95
353	3	1, 11, 6, 38, 327, 56, 309, 58, 307, 98, 267, 110, 255, 127, 238, 149, 216, 162, 203, 167, 198	94
359	7	14, 44, 343, 45, 342, 104, 283, 105, 282, 127, 260, 164, 223	96
367	3	3, 9, 6, 25, 354, 46, 333, 56, 323, 61, 318, 68, 311, 87, 292, 90, 289, 113, 266, 122, 257, 145, 234, 152, 227, 160, 219, 176, 203, 181, 198	92
373	2	4, 39, 342, 41, 340, 44, 337, 84, 297, 106, 275, 127, 254, 128, 253, 143, 238, 155, 226, 175, 206, 179, 202	94
379	2	4, 16, 371, 22, 365, 25, 362, 34, 353, 39, 348, 55, 332, 61, 326, 62, 325, 69, 318, 74, 313, 77, 310, 80, 307, 92, 295, 102, 285, 113, 274, 119, 268, 141, 246, 146, 241, 175, 212, 185, 202, 188, 199	89
383	5	1, 19, 10, 28, 375, 45, 358, 90, 313, 95, 308, 106, 297, 115, 288, 121, 282, 142, 261, 143, 260, 154, 249, 164, 239, 201, 202	93
389	2	4, 20, 377, 41, 356, 66, 331, 120, 277, 129, 268, 142, 255, 152, 245, 167, 230, 184, 213	95
397	2	4, 35, 370, 54, 351, 80, 325, 89, 316, 102, 303, 114, 291, 119, 286, 145, 260, 146, 259, 150, 255, 167, 238, 170, 235, 187, 218, 191, 214	93
401	3	6, 29, 384, 39, 374, 67, 346, 83, 330, 100, 313, 135, 278, 139, 274, 150, 263, 153, 260, 175, 238, 193, 220	94
409	7	6, 22, 13, 15, 14, 41, 396, 60, 377, 61, 376, 68, 369, 81, 356, 90, 347, 109, 328, 126, 311, 145, 292, 146, 291, 154, 283, 169, 268, 183, 254, 210, 227	92
419	2	4, 28, 399, 43, 384, 60, 367, 63, 364, 73, 354, 85, 342, 125, 302, 155, 272, 162, 265, 163, 264, 175, 252, 188, 239, 198, 229	94
421	2	4, 25, 404, 29, 400, 42, 387, 67, 362, 70, 359, 76, 353, 114, 315, 118, 311, 129, 300, 136, 293, 149, 280, 158, 271, 169, 260, 190, 239, 205, 224	93
431	7	2, 26, 14, 30, 429, 56, 403, 62, 397, 72, 387, 73, 386, 101, 358, 125, 334, 126, 333, 131, 328, 179, 280	95
433	5	6, 14, 10, 32, 421, 44, 409, 74, 379, 94, 359, 97, 356, 122, 331, 133, 320, 139, 314, 170, 283, 203, 250, 207, 246, 214, 239	94

(*continued on next page*)

Table 2 (*continued*)

p	δ	$a \in \mathbb{F}_p \backslash \{0, 4\delta\}$ which do not satisfy (∗)	%
439	3	6, 80, 371, 84, 367, 100, 351, 129, 322, 131, 320, 164, 287, 172, 279, 202, 249, 206, 245, 207, 244, 219, 232	95
443	2	1, 4, 22, 429, 25, 426, 40, 411, 44, 407, 50, 401, 56, 395, 75, 376, 82, 369, 83, 368, 84, 367, 91, 360, 94, 357, 102, 349, 110, 341, 120, 331, 138, 313, 154, 297, 214, 216, 235	91
449	3	1, 11, 6, 19, 442, 20, 441, 26, 435, 43, 418, 44, 417, 63, 398, 68, 393, 75, 386, 106, 355, 157, 304, 161, 300, 176, 285, 194, 267, 207, 254	93
457	5	10, 23, 454, 36, 441, 42, 435, 111, 366, 137, 340, 140, 337, 143, 334, 145, 332, 153, 324, 182, 295, 194, 283, 203, 274	95
461	2	4, 20, 449, 33, 436, 36, 433, 77, 392, 107, 362, 116, 353, 121, 348, 128, 341, 188, 281, 197, 272, 203, 266, 214, 255	95
463	3	1, 11, 6, 59, 416, 62, 413, 78, 397, 139, 336, 162, 313, 220, 255	97
467	2	4, 53, 422, 74, 401, 90, 385, 121, 354, 137, 338, 140, 335, 153, 322, 170, 305, 186, 289	96
479	13	26, 64, 467, 83, 448, 85, 446, 105, 426, 148, 383, 173, 358, 177, 354, 192, 339, 201, 330, 210, 321, 232, 299, 246, 285	95
487	3	6, 21, 478, 31, 468, 36, 463, 61, 438, 66, 433, 105, 394, 115, 384, 126, 373, 174, 325, 202, 297, 236, 263	95
491	2	4, 24, 475, 34, 465, 50, 449, 64, 435, 78, 421, 107, 392, 142, 357, 160, 339, 171, 328, 241, 258	96
499	2	3, 5, 4, 12, 495, 13, 494, 31, 476, 46, 461, 51, 456, 81, 426, 83, 424, 105, 402, 118, 389, 123, 384, 147, 360, 149, 358, 159, 348, 207, 300, 214, 293, 246, 261, 247, 260	93

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References

[1] E. Bannai, Character tables of commutative association schemes, in: W.M. Kantor et al. (Eds.), Finite Geometries, Buildings, and Related Topics, Clarendon Press, Oxford, 1990, pp. 105–128.

[2] E. Bannai, T. Ito, Algebraic Combinatorics I, Benjamin, Cummings, Menlo Park, CA, 1984.

[3] E. Bannai, T. Ito, The study of distance-regular graphs from the algebraic (i.e., character theoretical) viewpoint, in: Proceedings of Symposia in Pure Mathematics, vol. 47, American Mathematical Society, Providence, RI, 1987, pp. 343–349.

[4] E. Bannai, T. Ito, On distance-regular graphs with fixed valency, IV, European J. Combin. 10 (1989) 137–148.

[5] E. Bannai, S. Hao, S.-Y. Song, Character tables of the association schemes of finite orthogonal groups acting on the nonisotropic points, J. Combin. Theory Ser. A 54 (1990) 164–200.

[6] E. Bannai, S. Hao, S.-Y. Song, H. Wei, Character tables of certain association schemes coming from finite unitary and symplectic groups, J. Algebra 144 (1991) 189–213.

[7] E. Bannai, W.M. Kwok, S.-Y. Song, Ennola type dualities in the character tables of some association schemes, Mem. Fac. Sci. Kyushu Univ. Ser. A 44 (1990) 129–143.

[8] E. Bannai, O. Shimabukuro, H. Tanaka, Finite Euclidean graphs and Ramanujan graphs, Discrete. Math. (to appear).

[9] R. Evans, Character sums as orthogonal eigenfunctions of adjacency operators for Cayley graphs, in: Contemporary Mathematics, vol. 168, American Mathematical Society, Providence, RI, 1994, pp. 33–50.

[10] R. Evans, Spherical functions for finite upper half planes with characteristic 2, Finite Fields Appl. 1 (1995) 376–394.

[11] N.M. Katz, Estimates for Soto-Andrade sums, J. Reine Angew. Math. 438 (1993) 143–161.
[12] N.M. Katz, A note on exponential sums, Finite Fields Appl. 1 (1995) 395–398.
[13] W.M. Kwok, Character table of a controlling association scheme defined by the general orthogonal group $O(3, q)$, Graphs Combin. 7 (1991) 39–52.
[14] W.-C.W. Li, Number Theory with Applications, World Scientific, River Edge, NJ, 1996.
[15] A. Lubotzky, Discrete groups, Expanding Graphs and Invariant Measures, Birkhäuser Verlag, Basel, 1994.
[16] J. Soto-Andrade, Geometrical Gel' fand models, tensor quotients, and Weil representations, in: Proceedings of Symposia in Pure Mathematics, vol. 47, American Mathematical Society, Providence, RI, 1987, pp. 305–316.
[17] H. Tanaka, On some relationships among the association schemes of the orthogonal groups acting on hyperplanes, Master Thesis, Kyushu University, 2001.
[18] H. Tanaka, A four-class subscheme of the association scheme coming from the action of $PGL(2, 2^f)$, European J. Combin. 23 (2002) 121–129.
[19] A. Terras, Fourier Analysis on Finite Groups and Applications, London Mathematical Society Student Texts, vol. 43, Cambridge University Press, 1999.
[20] A. Weil, On the Riemann hypothesis in functionfields, Proc. Natl. Acad. Sci. USA 27 (1941) 345–347.